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LINKS BETWEEN COFINITE PRIME IDEALS IN QUANTUM FUNCTION ALGEBRAS

BY

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ABSTRACT

We describe the skew primitive elements in a multiparameter enveloping algebra $U = U_{q,p^{-1}}(g)$ and the links between cofinite maximal ideals in the corresponding quantum function algebra $\mathbb{C}_q[G]$. These results are applied to determine the coradical filtration for U, and to obtain a moduli space for multiparameter Drinfeld doubles.

Introduction

In the theory of quantum groups we encounter two kinds of Hopf algebras each of which may be regarded as a dual of the other. Let G be a connected simple algebraic group over $\mathbb{C}$ with Lie algebra g, $U_q(g)$ the quantized enveloping algebra of g and $\mathbb{C}_q[G]$ the quantum function algebra of G. The duality referred to above consists of a pairing $\mathbb{C}_q[G] \times U_q(g) \longrightarrow \mathbb{C}$ of Hopf algebras which is nondegenerate in each variable.

We can obtain a multiparameter version of this pairing in the following way. First $U_q(g)$ is a quotient of the Drinfeld double $D_q(g)$ of the quantized enveloping algebra of a Borel subalgebra of g. The definitions of $\mathbb{C}_q[G]$ and $D_q(g)$ involve the character group L of G, and both of these Hopf algebras are graded by $L \times L$.

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If $p \in H^2(L, \mathbb{C}^*)$, then following [AST] and [HLT] we form the twisted Hopf algebras $A = \mathbb{C}_{q,p}[G]$ and $D = D_{q,p^{-1}}(g)$. Again there is a pairing $A \times D \longrightarrow \mathbb{C}$, but this is no longer nondegenerate in the second variable. Accordingly we denote the radical of this pairing in D by R and set $U = U_{q,p^{-1}}(g) = D/R$, the multiparameter quantized enveloping algebra. In this paper we study an important aspect of the duality between A and U, namely the relationship of links between maximal ideals of finite codimension in A to skew primitive elements of U.

In more detail, we start by recalling the construction of A and D in Section 1. We also show that R is the augmentation ideal of a certain group of central group-like elements in D. We describe this group explicitly and establish some of its properties.

In the next section, which is the heart of the paper, we determine the links between cofinite maximal ideals of A and the multiplicities of these links. There is some overlap here with the work in [BG] on cliques of prime ideals in quantum function algebras, see 2.2 for details. By duality we obtain a description of the skew primitive elements of U.

As an application of this work we determine the coradical filtrations of D and U. We also describe the Hopf ideals of D and the Hopf algebra maps between multiparameter Drinfeld doubles. In the one-parameter case results related to these are obtained in [B], [C], [CM 3] and [T], for comments see 3.5. We also obtain a moduli space for the Drinfeld doubles with q, G fixed. To describe this result, assume here that G is not of type $D_{2\ell}, \ell \geq 2$ (some minor modifications are necessary in this case). Let Γ be the automorphism group of the Dynkin diagram of g. Then Γ acts on $H^2(L, \mathbb{C}^*)$ and the quotient variety $H^2(L, \mathbb{C}^*)/\Gamma$ is the required moduli space. In addition the group of Hopf algebra automorphisms of $U_{q,p^{-1}}(g)$ has the form $N\Gamma_p$, where N is the group of diagonal automorphisms, and Γ_p is the stabilizer of p in Γ.

Until section 3.6, the base field is $\mathbb{C}$ and q is a nonzero complex number which is not a root of unity. We occasionally quote some results from [J] and [JL] in which q is an indeterminate. However, these results are still valid when q is not a root of unity. This can be shown by specializing q, see [J, 10.5.2] for some discussion.

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1. Preliminaries

1.1. Let g be a complex simple Lie algebra of rank n with Cartan matrix (a_{ij}). There are relatively prime positive integers $\{d_i\}$ such that $(d_i a_{ij})$ is symmetric.

Let h be a Cartan subalgebra of g and $\{\alpha_1, \dots, \alpha_n\}$ a basis for the corresponding root system. We denote the lattices of weights and roots of g by P and Q respectively. Let $\overline{\omega}_1, \dots, \overline{\omega}_n$ be the fundamental weights and $P^+ = \sum_{i=1}^n \mathbb{N}\overline{\omega}_i$. We can define a nondegenerate bilinear form (,) on h^* such that

$$(\overline{\omega}_i, \alpha_j) = \delta_{ij} d_i, \quad (\alpha_i, \alpha_j) = d_i a_{ij}$$

for all i, j. Note that these equations imply that $\alpha_i = \sum_j (d_i a_{ij}/d_j)\overline{\omega}_j$. Let $h_1, \dots, h_n$ be the basis of h be such that $\overline{\omega}_i(h_j) = \delta_{ij}$.

1.2. We recall some results from [HLT] on deformations of bigraded Hopf algebras.

Let $u \in \Lambda^2 h$ and write $u = \sum u_{ij} h_i \otimes h_j$ for a skew symmetric matrix (u_{ij}). We can view u either as an alternating bilinear form on h^* or as the linear map $u \in \operatorname{End} h$ given by

$$u(x) = \sum_{i,j} u_{ij}(x, h_i)h_j \quad \text{for } x \in h.$$

Let ${}^t u \in \operatorname{End} h^*$ be the transpose of $u, \Phi = -{}^t u$ and $\Phi_\pm = \Phi \pm I$. It is easy to show that

$$u(\lambda, \mu) = (\Phi\lambda, \mu) \quad \text{for all } \lambda, \mu \in h^*.$$

Now suppose $\hbar \in \mathbb{C}^* \backslash i\pi\mathbb{Q}$ and set $q = \exp(-\hbar/2)$ and

$$p(\lambda, \mu) = q^{\frac{1}{2}u(\lambda,\mu)} = \exp(-\hbar u(\lambda, \mu)/4) \quad \text{for } \lambda, \mu \in h^*.$$

Then $p(\, , \,)$ is an antisymmetric bicharacter on h^*.

Suppose that L is a lattice in h^* and that $A = \bigoplus_{(\lambda,\mu)\in L\times L} A_{\lambda,\mu}$ is a L-bigraded Hopf algebra as in [HLT,2.1]. We can define a new L-bigraded Hopf algebra A_p which is equal to A as an L-bigraded coalgebra, has the same antipode as A, and has multiplication given by

$$a.b = p(\lambda, \lambda')p(\mu, \mu')^{-1}ab$$

for all $a \in A_{\lambda,\mu}, b \in A_{\lambda',\mu'}$.

Next let A and U be L-bigraded Hopf algebras and $\langle | \rangle \colon A \times U \to \mathbb{C}$ a bilinear pairing of Hopf algebras as in [HLT, 2.3]. We assume that

$$\langle A_{\lambda,\mu} | U_{\gamma,\delta} \rangle = 0 \quad \text{if } \lambda + \mu \neq \gamma + \delta.$$

The pair $\{A, U\}$ is called an L-bigraded pair. By [HLT, Theorem 2.4] $\{A_{p^{-1}}, U_p\}$ is an L-bigraded pair with pairing $\langle | \rangle_p$ given by

$$\langle a_{\lambda,\mu} | u_{\gamma,\delta} \rangle_p = p(\lambda,\gamma)^{-1} p(\mu,\delta)^{-1} \langle a_{\lambda,\mu} | u_{\gamma,\delta} \rangle.$$

Now suppose that $\{A^{op}, U\}$ is an L-bigraded pair. We can form the Drinfeld double $A \bowtie U$ as in [HLT, 2.3]. By [HLT, Theorem 2.6] $A \bowtie U$ is an L-bigraded Hopf algebra and there is a natural isomorphism of L-bigraded Hopf algebras

$$(A \bowtie U)_p \cong A_p \bowtie U_p.$$

1.3. Let G be the connected simple algebraic group with maximal torus H such that $Lie(G) = g$ and $X(H) = L$. For $1 \leq i \leq n$ set $q_i = q^{d_i}, \widehat{q}_i = (q_i - q_i^{-1})^{-1}$.

Let U^0 be the group algebra of $X(H)$. Thus

$$U^0 = \mathbb{C}[k_\lambda; \lambda \in L], \quad k_0 = 1, \quad k_\lambda k_\mu = k_{\lambda+\mu}.$$

We set $k_i = k_{\alpha_i}$. The one-parameter quantized enveloping algebra is the Hopf algebra

$$U_q(g) = U^0[e_i, f_i; 1 \leq i \leq n]$$

with defining relations:

$$k_\lambda e_j k_\lambda^{-1} = q^{(\lambda,\alpha_j)} e_j,$$
$$k_\lambda f_j k_\lambda^{-1} = q^{-(\lambda,\alpha_j)} f_j,$$
$$e_i f_j - f_j e_i = \delta_{ij} \widehat{q}_i (k_i - k_i^{-1}),$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} e_i^{1-a_{ij}-k} e_j e_i^k = 0, \quad \text{if } i \neq j,$$
$$\sum_{k=0}^{1-a_{ij}} (-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_{q_i} f_i^{1-a_{ij}-k} f_j f_i^k = 0, \quad \text{if } i \neq j,$$

where $[m]_t = (t - t^{-1}) \cdots (t^m - t^{-m})$ and

$$\begin{bmatrix} m \\ k \end{bmatrix}_t = \frac{[m]_t}{[k]_t [m-k]_t}.$$

The Hopf algebra structure is given by

$$\Delta(k_\lambda) = k_\lambda \otimes k_\lambda, \quad \epsilon(k_\lambda) = 1, \quad S(k_\lambda) = k_\lambda^{-1},$$
$$\Delta(e_i) = e_i \otimes 1 + k_i \otimes e_i, \quad \Delta(f_i) = f_i \otimes k_i^{-1} + 1 \otimes f_i,$$
$$\epsilon(e_i) = \epsilon(f_i) = 0, \quad S(e_i) = -k_i^{-1}e_i, \quad S(f_i) = -f_i k_i.$$

We denote by $U_q(b^+)$ (resp. $U_q(b^-)$) the subalgebra of $U_q(g)$ generated by $e_1, \dots, e_n$ and $k_\lambda, \lambda \in L$ (resp. $f_1, \dots, f_n$ and $k_\lambda, \lambda \in L$).

1.4. We now consider the Drinfeld double

$$D_q(g) = U_q(b^+) \bowtie U_q(b^-)$$

associated to the Rosso–Tanisaki–Killing form

$$\langle | \rangle \colon U_q(b^+) \otimes U_q(b^-) \longrightarrow \mathbb{C},$$

see [HLT, 3.2] for more details. Thus

$$D_q(g) = \mathbb{C}[s_\lambda, t_\lambda, e_i, f_i;\ \lambda \in L,\ 1 \le i \le n]$$

where $s_\lambda = k_\lambda \otimes 1,\ t_\lambda = 1 \otimes k_\lambda,\ e_i = e_i \otimes 1,\ f_i = 1 \otimes f_i$. There is an isomorphism

$$D_q(g)/(s_\lambda - t_\lambda; \lambda \in L) \cong U_q(g)$$

given by

$$e_i \longrightarrow e_i, \quad f_i \longrightarrow f_i, \quad s_\lambda \longrightarrow k_\lambda, \quad t_\lambda \longrightarrow k_\lambda.$$

By [HLT, Corollary 3.3] $\{U_q(b^+)^{op}, U_q(b^-)\}$ is an L-bigraded dual pair with

$$k_\lambda \in U_q(b^\pm)_{-\lambda,\lambda},\ e_i \in U_q(b^+)_{-\alpha_i,0}, \quad f_i \in U_q(b^-)_{0,-\alpha_i};$$

in addition $D_q(g)$ is an L-bigraded Hopf algebra with

$$s_\lambda \in D_q(g)_{-\lambda,\lambda}, \quad t_\lambda \in D_q(g)_{\lambda,-\lambda},$$
$$e_i \in D_q(g)_{-\alpha_i,0}, \quad f_i \in D_q(g)_{0,\alpha_i}.$$

We have

$$\Delta e_i = e_i \otimes 1 + s_i \otimes e_i$$

and

$$\Delta f_i = f_i \otimes t_i^{-1} + 1 \otimes f_i.$$

1.5. Let M be a left $U_q(g)$-module. We say an element $x \in M$ has weight $\mu \in L$ if $k_\lambda x = q^{(\lambda,\mu)}x$ for all $\lambda \in L$, and denote the subspace of elements of weight μ by M_μ. Set $L^+ = L \cap P^+$. It is well known that for all $\Lambda \in L^+$ there exists a unique finite-dimensional simple $U_q(g)$-module $L(\Lambda)$ with highest weight Λ, and lowest weight $w_0\Lambda$ where w_0 is the longest element of the Weyl group W. If $L(\Lambda)_\mu$ is one dimensional, choose v_μ such that $L(\Lambda)_\mu = \mathbb{C}v_\mu$.

We denote the category of finite-dimensional $U_q(g)$-modules which are direct sums of the modules $L(\Lambda)$ by $\mathcal{C}_q$. The category $\mathcal{C}_q$ is closed under the formation of tensor products and duals.

For $M \in \mathrm{obj}(\mathcal{C}_q)$, $f \in M^*$, $v \in M$ we define the function $c_{f,v} \in U_q(g)^0$ by

$$c_{f,v}(u) = f(uv) \quad \text{for } u \in U_q(g).$$

The quantized function algebra $\mathbb{C}_q[G]$ is defined as the dual of $U_q(g)$ with respect to the category $\mathcal{C}_q$, that is

$$\mathbb{C}_q[G] = \mathbb{C}[c_{f,v}; v \in M, f \in M^*, M \in \mathrm{obj}(\mathcal{C}_q)].$$

Then $\mathbb{C}_q[G]$ is an L-bigraded Hopf algebra with

$$\mathbb{C}_q[G]_{\lambda,\mu} = \mathrm{span}\{c_{f,v}; v \in M_\mu, f \in M^*_\lambda, M \in \mathrm{obj}(\mathcal{C}_q)\}.$$

Since $\mathbb{C}_q[G] \subseteq U_q(g)^0$, there is a duality pairing

$$\langle | \rangle \colon \mathbb{C}_q[G] \times D_q(g) \longrightarrow \mathbb{C}$$

and this makes $\{\mathbb{C}_q[G], D_q(g)\}$ into an L-bigraded dual pair [HLT, Theorem 3.4].

1.6. We now apply the twisting procedure outlined in 1.2 to the L-bigraded pair $\{U_q(b^+)^{op}, U_q(b^-)\}$. We call

$$D_{q,p^{-1}}(g) = (U_q(b^+) \bowtie U_q(b^-))_{p^{-1}} \cong U_{q,p^{-1}}(b^+) \bowtie U_{q,p^{-1}}(b^-)$$

the multiparameter Drinfeld double. As a $\mathbb{C}$-algebra it is generated by elements $e_i = e_i \otimes 1$, $f_i = 1 \otimes f_i$, $s_\lambda = k_\lambda \otimes 1$, $t_\lambda = 1 \otimes k_\lambda, \lambda \in L$, $1 \le i \le n$. These elements satisfy the relations

$$\begin{aligned} s_\lambda t_\mu &= t_\mu s_\lambda, \\ e_i f_j - f_j e_i &= \delta_{ij}\widehat{q}_i(s_{\alpha_i} - t_{\alpha_i}^{-1}), \\ s_\lambda e_j s_{-\lambda} &= q^{-(\Phi_-\lambda,\alpha_j)} e_j, \\ s_\lambda f_j s_{-\lambda} &= q^{(\Phi_-\lambda,\alpha_j)} f_j, \\ t_\lambda e_j t_{-\lambda} &= q^{(\Phi_+\lambda,\alpha_j)} e_j, \\ t_\lambda f_j t_{-\lambda} &= q^{-(\Phi_+\lambda,\alpha_j)} f_j. \end{aligned}$$

These relations are given in the one-parameter case (i.e. when $p = 1$) in [HLT, 3.2]. In general they can be obtained by the twisting process described in 1.2.

The other relations, known as the Serre relations, can be conveniently expressed in terms of various adjoint actions of $D = D_{q,p^{-1}}(g)$. Let S be the antipode of D and S' the composition inverse of S. For $a, b \in D$ we set

$$\begin{aligned} ad_r(a)(b) &= \sum S(a_{(1)})ba_{(2)}, \\ ad'_r(a)(b) &= \sum S'(a_{(2)})ba_{(1)}. \end{aligned}$$

For example $ad'_r(e_i)(b) = be_i - e_i s_i^{-1} b s_i$. Using this and induction we obtain in the one-parameter case

$$(ad'_r(e_i))^n(e_j) = \sum_{k=0}^{n} (-1)^k q_i^{(1-a_{ij}-n)k} \begin{bmatrix} n \\ k \end{bmatrix}_{q_i} e_i^k e_j e_i^{n-k}$$

and hence $(ad'_r(e_i))^{1-a_{ij}}(e_j) = 0$ for all i, j. Using [CM2, Lemma 3.2], it follows that this relation holds also in the multiparameter case. Similarly, the Serre relation for f_i, f_j may be written in the form $ad_r(f_i))^{1-a_{ij}}(f_j) = 0$.

Remark: Two additional adjoint actions ad_l and ad'_l are defined in [CM2]. In the Serre relations above we may substitute $ad_l(e_i)$ for $ad'_r(e_i)$ and $ad'_l(f_i)$ for $ad_r(f_i)$.

1.7. We now consider the L-bigraded dual pair $\{\mathbb{C}_{q,p}[G], D_{q,p^{-1}}(g)\}$ obtained by twisting the pair $\{\mathbb{C}_q[G], D_q(g)\}$. The twisted pairing $\langle|\rangle_p$ is given by $\langle a|u\rangle_p = p(\lambda,\gamma)p(\mu,\delta)\langle a|u\rangle$ for all $a \in \mathbb{C}_{q,p}[G]_{\lambda,\mu}$, $u \in D_{q,p^{-1}}(g)_{\gamma,\delta}$ [HLT, Theorem 3.6]

If M is an object in the category $\mathcal{C}_q$ of left $D_q(g)$-modules defined in 1.5, we can make M into a left $D_{q,p^{-1}}(g)$-module by defining

$$u.x = p(\lambda, \delta - \gamma)p(\delta, \gamma)ux$$

for all $u \in D_q(g)_{\gamma,\delta}, x \in M_\lambda$. By [HLT, Theorem 3.8] $\mathbb{C}_{q,p}[G]$ may be identified with the Hopf algebra of coordinate functions on $C_{q,p}$, that is we have for all $f \in M^*, v \in M$ and $u \in D_{q,p^{-1}}(g)$ that

$$\langle f|u.v\rangle = \langle c_{f,v}|u\rangle_p.$$

1.8. We introduce some further notation. Let R be the radical of the pairing $\langle , \rangle_p$ in $D = D_{q,p^{-1}}(g)$, that is $R = \{d \in D| \langle \ , d\rangle_p = 0\}$. Since $\langle , \rangle_p$ is a pairing of Hopf algebras, R is a Hopf ideal of D. Later we refer to R as the radical of D. We set $U = U_{q,p^{-1}}(g) = D/R$ and call U the multiparameter (quantized) enveloping algebra. Let D^+ (resp. D^-) be the subalgebra of D generated by the elements e_i (resp. f_i), $1 \le i \le n$ and let D^0 be the group algebra of the group $T = \{s_\lambda t_\mu, \lambda, \mu \in L\}$. We denote the images of $D^\pm$ and D^0 in U by $U^\pm$ and U^0.

There is a triangular decomposition $D = D^- \otimes D^0 \otimes D^+$. If Z is any subgroup of $T, \nabla(Z)$ denotes the augmentation ideal of Z in D, that is the two-sided ideal of D generated by the elements $z-1, z \in Z$. If M is any bi-ideal of D, we set $M^\dagger = \{g \in T| g-1 \in M\}$.

We define

$$Z_L = \{s_\lambda t_\mu \in T|(\Phi_-\lambda - \Phi_+\mu, \beta) \in \frac{4\pi i\mathbb{Z}}{\hbar} \text{ for all } \beta \in L\}.$$

Clearly $Z_L \subseteq Z_Q$. It follows from the relations in 1.6 that Z_Q is the intersection of T with the center of D.

Note that D can be made into a Q-graded algebra by setting $D_\alpha = \sum_{\gamma+\delta=\alpha} D_{\gamma,\delta}$ for $\alpha \in Q$. We have $s_\lambda, t_\mu \in D_0$, $e_i \in D_{-\alpha_i}$ and $f_i \in D_{\alpha_i}$. For $\lambda \in L$ we have $s_\lambda t_\lambda e_i s_{-\lambda} t_{-\lambda} = q^{(\Phi_+\lambda - \Phi_-\lambda, \alpha_i)} e_i = q^{2(\lambda,\alpha_i)} e_i$. It follows that $D_\alpha = \{x \in D| s_\lambda t_\lambda x s_{-\lambda} t_{-\lambda} = q^{-(2\lambda,\alpha)} x\}$, that is the Q-grading on D is determined by the action of the elements $s_\lambda t_\lambda$ by conjugation. Therefore any ideal of D is homogeneous for the Q-grading, and any factor algebra of D is Q-graded.

We can now characterize the radical R.

THEOREM: *With the above notation* $R = \nabla(Z_L)$.

Proof: If $\alpha = \sum n_i \alpha_i \in Q$ we define $|\alpha| = \sum n_i$. Then setting $U_m = \sum_{|\alpha|=m} U_\alpha$ makes U into a $\mathbb{Z}$-graded Hopf algebra. It follows from [R, Theorem 3] that U has a triangular decomposition

$$U = U^- \otimes U^0 \otimes U^+.$$

Next we claim that $D^+ \cap R = 0$. Set $D^+_\alpha = D^+ \cap D_\alpha$ where $D = \oplus D_\alpha$ is the Q-grading defined above. It suffices to show $D^+_\alpha \cap R = 0$ for all $\alpha \in Q$. By the nature of the grading on D^+, if $d \in D^+_\alpha$, then $d \in D_{\alpha,0}$ and we have for all $a \in \mathbb{C}_{q,p}[G]_{\lambda,\mu}$ that

$$\langle a|d\rangle_p = p(\lambda, \alpha)\langle a|d\rangle.$$

Thus the assertion reduces to the 1-parameter case. In this case $D_q(g)/(s_\lambda - t_\lambda) \cong U_q(g)$, and we claim the induced pairing $\langle | \rangle: \mathbb{C}_q[G] \times U_q(g) \to \mathbb{C}$ is nondegenerate, in $U_q(g)$. In fact $\langle |u\rangle = 0$ if and only if $u \in \cap \operatorname{ann} L(\Lambda)$ for all $\Lambda \in L^+$. However by [J, Lemma 7.1.9] or [JL, Lemma 8.3] this intersection is zero. The claim follows easily. Similarly $D^- \cap R = 0$.

Now let $\{u_i\}$, (resp. $\{w_i\}$) be $\mathbb{C}$ bases of D^- (resp. D^+). If $\sum u_i \otimes v_{ij} \otimes w_j \in R$ with $v_{ij} \in D^0$ then since $D^\pm$ map isomorphically onto $U^\pm$ we have $v_{ij} \in R$ for all i, j. Thus R is generated by its intersection with D^0.

Since $R \cap D^0$ is a Hopf ideal of the group algebra $D^0 = \mathbb{C}[s_\lambda t_\mu \colon \lambda, \mu \in L]$ it is equal to the augmentation ideal of the subgroup $\{s_\lambda t_\mu \colon \langle , s_\lambda t_\mu\rangle_p = 1\}$. Now $\langle , s_\lambda t_\mu\rangle_p = 1$ if and only if $f(s_\lambda t_\mu . v) = 1$ for all $M \in \operatorname{obj}(\mathcal{C}_{q,p}), v \in M_\beta$ and $f \in M^*_\beta$ such that $f(v) = 1$. We have

$$\begin{aligned} f(s_\lambda t_\mu . v) &= p(\beta, 2\lambda - 2\mu) f(s_\lambda t_\mu v) \\ &= q^{u(\mu-\lambda,\beta)+(\lambda+\mu,\beta)} \\ &= q^{(\Phi_+\mu - \Phi_-\lambda,\beta)} \end{aligned}$$

It follows that $\{s_\lambda t_\mu \colon \langle\ , s_\lambda t_\mu\rangle_p = 1\} = Z_L$.

Remark: For later use we record the fact that for $v \in M_\beta, f \in M^*_{-\beta}$ such that $f(v) = 1$, we have

$$t_i(c_{f,v}) = q^{(\Phi_+\alpha_i,\beta)}$$

and

$$s_i(c_{f,v}) = q^{-(\Phi_-\alpha_i,\beta)}.$$

1.9. PROPOSITION: *If $s_\lambda t_\mu \in Z_L$ then $(\lambda, \lambda) = (\mu, \mu)$*

Proof: If $\lambda \in L$ and $\lambda = \sum_{i=1}^n \lambda_i \alpha_i$ with $\lambda_i \in \mathbb{Q}$, we set $\underline{\lambda} = (\lambda_1 d_1, \ldots, \lambda_n d_n) \in \mathbb{Q}^n$. Let B be the symmetric matrix $B = (a_{ij}/d_j)$. Then $(\lambda, \mu) = \underline{\lambda} B \underline{\mu}^t$. Now let $c_{ij} = (\Phi\overline{\omega}_i, \overline{\omega}_j)$ and $C = (c_{ij})$. Then $(\Phi_\pm\mu, \overline{\omega}_k)$ is the k^{th} entry of the vector

$\underline{\mu}(BC \pm 1)$. There exists a positive integer N such that $N\overline{\omega}_k \in L$ for $k = 1, \ldots, n$. Thus if $s_\lambda t_\mu \in Z_L$ we have

$$\underline{\lambda} + \underline{\mu} \equiv (\underline{\lambda} - \underline{\mu})BC \mod \frac{\pi i \mathbb{Q}}{\hbar}.$$

Since BCB is skew symmetric, this implies

$$\begin{aligned}(\lambda + \mu)B(\lambda - \mu)^t &\equiv (\underline{\lambda} - \mu)BCB(\underline{\lambda} - \underline{\mu})^t \\ &\equiv 0 \mod \frac{\pi i \mathbb{Q}}{\hbar}.\end{aligned}$$

However $(\underline{\lambda} + \underline{\mu})B(\underline{\lambda} - \underline{\mu})^t \in \mathbb{Q}$ and $\hbar \notin \pi i \mathbb{Q}$, so $\underline{\mu}B\underline{\mu}^t = \underline{\lambda}B\underline{\lambda}^t$.

An analogue of the next result for $g = gl(n)$ is proved in [CM2, Lemma 4.2].

COROLLARY: *We have*

$$\{\lambda \in L | s_\lambda \in Z_L\} = 0 = \{\mu \in L | t_\mu \in Z_L\}$$

Proof: This follows since by [H, 8.5] the bilinear form (,) is positive definite on $L \otimes_{\mathbb{Z}} \mathbb{Q}$.

2. Links between cofinite primes of $\mathbb{C}_{q,p}[G]$

2.1. Let $A = \mathbb{C}_{q,p}[G]$, $U = U_{q,p^{-1}}(g)$ and define ideals $I^\pm$ of A by

$$I^+ = \langle c_{f,v_\Lambda} | f \in (U_{q,p^{-1}}(b^+)L(\Lambda)_\Lambda)^\perp \rangle$$

and

$$I^- = \langle c_{f,v_{w_0\Lambda}} | f \in (U_{q,p^{-1}}(b^-)L(\Lambda)_{w_0\Lambda})^\perp \rangle.$$

The ideals $I^\pm$ are denoted $I_e^\pm$ in [HLT]. We set $I = I^+ + I^-$.

Let H be a maximal torus of G with coordinate ring $\mathbb{C}[H]$. There is an algebra map φ: $A \longrightarrow \mathbb{C}[H], c \longmapsto \varphi_c$ defined by $\varphi_c(h) = \chi_h(c)$, where χ_h is the one-dimensional representation of A determined by $\chi_h(c_{g,v}) = \mu(h)g(v)$ for $v \in L(\Lambda)_\mu$ and $\mu \in L(\Lambda)^*$. Thus if $g(v) = 1$, the image of $c_{g,v}$ under φ is the character μ of H, which is a unit in $\mathbb{C}[H]$.

Finally we note that all finite-dimensional simple A-modules are one dimensional and annihilated by I, and that $\ker\varphi = I$, so that φ defines an isomorphism $A/I \cong \mathbb{C}[H]$. This is shown in the 1-parameter case in [J, Lemma 9.3.11] and a similar proof works in general.

2.2. If H is a bialgebra, we denote the dual of H by $H^\star$, as in [J, 1.4] and the set of group-like elements of H by $G(H)$. By the preceding remarks any maximal ideal of finite codimension in A has codimension one, and thus has the form $m_\chi = \ker\chi$ for some group-like element χ of $A^\star$.

Let m and m' be codimension one ideals of A. We say that m' is linked to m and write $m' \rightsquigarrow m$ if $m'm$ is strictly contained in $m \cap m'$. In this case the dimension of $(m\cap m')/m'm$ as a k-vector space is called the multiplicity of the link from m' to m and denoted $\mathrm{mult}(m', m)$. A more general definition of a link between prime ideals of a Noetherian ring and of the multiplicity of a link is given in [Jat]. The connected component of the graph of links of A containing P is called the clique of P. In the one-parameter case the cliques of arbitrary prime ideals of A are described in [BG, Theorem 0.5] in terms of certain subgroups of $\mathrm{Aut}(A)$. In the case $G = \mathrm{SL}(n)$, the subgroup yielding the links between prime ideals of finite codimension, as well as the links themselves are described explicitly in [BG, Example 6.12]. On the other hand the issue of multiplicities, which will be crucial to us in Section 3, is not discussed in [BG].

We denote the image of the group-like elements s_i, t_i of $D_q(g)$ in U by the same symbol. Some of these group-likes may become equal in U. Thus we regard $\Omega = \{s_1, \dots, s_n, t_1, \dots, t_n\} \subseteq U \subseteq A^\star$ as a set with multiplicities. We can now state the main result of this section.

THEOREM: *If $m_{\chi'} \rightsquigarrow m_\chi$ then one of the following holds:*
(a) $\chi = \chi'$ *and* $\mathrm{mult}(m_{\chi'}, m_\chi) = n$,
(b) $\chi'\chi^{-1} \in \Omega$ *and* $\mathrm{mult}(m_{\chi'}, m_\chi) = |\{g \in \Omega | \chi'\chi^{-1} = g\}|$.

2.3. We need a result giving commutation relations in A from [HLT]. Let C_β be the canonical element of $D_{q,p^{-1}}(g)$ associated to the nondegenerate pairing $U_{q,p^{-1}}(b^+)_{-\beta,0} \otimes U_{q,p^{-1}}(b^-)_{0,-\beta} \to \mathbb{C}$ and for $M, N \in \mathrm{obj}\, \mathcal{C}_{q,p}$ let $C\colon M \otimes N \to M \otimes N$ be the operator given by multiplication by $C = \sum_{\beta \in Q_+} C_\beta$.

LEMMA: *For $\Lambda, \Lambda' \in L^+$, let*

$$g \in L(\Lambda')^*_{-\eta}, \ f \in L(\Lambda)^*_{-\mu}, \quad v \in L(\Lambda')_\gamma, \quad v_\Lambda \in L(\Lambda)_\Lambda$$

and

$$v_{w_0\Lambda} \in L(\Lambda)_{w_0\Lambda}.$$

Then

(a)
$$c_{g,v}c_{f,v_\Lambda} = q^{(\Phi+\Lambda,\gamma)-(\Phi+\mu,\eta)}(c_{f,v_\Lambda}c_{g,v} + \sum_{\nu\in Q_+} c_{f_\nu,v_\Lambda}c_{g_\nu,v}),$$

where

$$f_\nu \in (U_{q,p^{-1}}(b^+)f)_{-\mu+\nu}$$

and

$$g_\nu \in (U_{q,p^{-1}}(b^-)g)_{-\eta-\nu}$$

are such that

$$\sum f_\nu \otimes g_\nu = \sum_{\beta\in Q^+\setminus\{0\}} C_\beta(f\otimes g);$$

(b)
$$c_{f,v_{w_0\Lambda}}c_{g,v} = q^{(\Phi-\mu,\eta)-(\Phi-w_0\Lambda,\gamma)}(c_{g,v}c_{f,v_{w_0\Lambda}} + \sum c_{g_\nu,v}c_{f_\nu,v_{w_0\Lambda}}),$$

where

$$f_\nu \in (U_{q,p^{-1}}(b^-)f)_{-\mu-\nu}$$

and

$$g_\nu \in (U_{q,p^{-1}}(b^+)g)_{-\eta+\nu}$$

are such that

$$\sum g_\nu \otimes f_\nu = \sum_{\beta\in Q^+\setminus\{0\}} C_\beta(g\otimes f).$$

Proof: See [HLT, Corollary 3.10] for (a); (b) is proved in a similar way.

2.4. We let J be the intersection of the annihilators of the two-dimensional A-modules. If m and m' are codimension one ideals of A, then since $m/m'm$ is a semisimple A/m'-module, we have $J \subseteq m'm$. Hence as far as the computation of $\operatorname{mult}(m', m)$ is concerned we may pass to the algebra A/J. Clearly $I^2 \subseteq J$.

PROPOSITION: *Let $\Lambda \in L^+$ and $g \in L(\Lambda)^*_{-\eta}$.*

(a) *If $\eta \notin \{\Lambda, \Lambda-\alpha_1, \ldots, \Lambda-\alpha_n\}$ then $c_{g,v_\Lambda} \in J$.*

(b) *If $\eta \notin \{w_0\Lambda, w_0\Lambda+\alpha_1, \ldots, w_0\Lambda+\alpha_n\}$ then $c_{g,v_{w_0\Lambda}} \in J$.*

Proof: The idea of the proof is to find a relation similar to 2.3(a) in which all terms except the one containing c_{g,v_Λ} belong to J. We can assume $g \neq 0$.

Since $L(\Lambda)^*$ has lowest weight $-\Lambda$ we have

$$\sum_{i=1}^n e_i L(\Lambda)^* = \sum_{\nu\neq-\Lambda} L(\Lambda)^*_\nu,$$

so we may assume $g = e_i f$ for some i and $f \in L(\Lambda)^*_{-\mu}$ where $\mu = \eta + \alpha_i$. Now by the representation theory of $U_q(\mathrm{sl}(2))$ applied to the algebra generated by e_i, f_i and $k_i^{\pm 1}$, $f_i g \neq 0$. Choose $v \in L(\Lambda)_\mu$ with $(f_i g)(v) \neq 0$.

We now consider the relation given in Lemma 2.3(a). Since $\mu \neq \Lambda$ and $\eta \neq \mu$ we have $f(v_\Lambda) = g(v) = 0$. Hence c_{f,v_Λ} and $c_{g,v}$ belong to I. If $\nu \in Q^+$ and $c_{g_\nu,v} \notin I$ we have $g_\nu(v) \neq 0$ and thus $\eta + \nu = \mu$, which gives $\nu = \alpha_i$. For a similar reason $c_{f_\nu,v_\Lambda} \in I$ for all ν. Now $C_{\alpha_i}(f \otimes g)$ is a scalar multiple of $e_i f \otimes f_i g$ so Lemma 2.3(a) gives $c_{e_i f, v_\Lambda} c_{f_i g, v} \in I^2$. However, since $f_i g(v) \neq 0$, $c_{f_i g, v}$ maps to a unit mod I and hence mod I^2. Since $g = e_i f$, we obtain $c_{g,v_\Lambda} = c_{e_i f, v_\Lambda} \in I^2$. This proves (a) and the proof of (b) is similar.

2.5. We examine the ideal I/J of A/J more closely. For $\Lambda \in L$ there exists a unique element $w \in W$ such that $w\Lambda \in L^+$. The module $L(w\Lambda)$ has Λ as a weight of multiplicity one. Choose $v \in L(w\Lambda)_\Lambda$ and $v^* \in L(w\Lambda)^*_{-\Lambda}$ such that $v^*(v) = 1$ and set $g_\Lambda = c_{v^*,v}$. We have $g_\Lambda(s_\lambda t_\mu) = q^{(\Phi_+ \mu - \Phi_- \lambda, \Lambda)}$ by the remark at the end of 1.8. Next for each simple root α_i and $\Lambda \in L^+$, $\dim L(\Lambda)_{\Lambda - \alpha_i} = 0$ or 1, since $\Lambda - \alpha_i$ occurs as a weight of the corresponding Verma module with multiplicity one. Also $L(\Lambda)_{\Lambda-\alpha_i} \neq 0$ if and only if $w = f_i v_\Lambda \neq 0$. In this case there is a unique element $w^* \in L(\Lambda)^*_{-\Lambda+\alpha_i}$ such that $w^*(w) = 1$ and we write $c_i^+(\Lambda)$ for c_{w^*,v_Λ}. Recall the Q-grading on U defined in 1.8. Set $U_+^+ = \bigoplus_{\beta \neq 0} U_\beta^+$ It is easy to verify that

$$c_i^+(\Lambda)(U^- \otimes U^0 \otimes U_+^+) = 0$$

$$c_i^+(\Lambda)(U^- \otimes U^0)_\beta = 0 \quad \text{if } \beta \neq \alpha_i$$

and

$$c_i^+(\Lambda)(f_i s_\lambda t_\mu) = q^{(\Phi_+ \mu - \Phi_- \lambda, \Lambda)},$$

Clearly $c_i^+(\Lambda)$ is uniquely determined by these conditions. Now by Proposition 2.4, I^+/J is generated by the images of the elements $c_i^+(\Lambda)$ for $1 \leq i \leq n$ and $\Lambda \in L^+$. Now for $\Lambda, \mu \in L^+$ such that $\Lambda - \alpha_i$ (resp. $\mu - \alpha_i$) is a weight of $L(\Lambda)$ (resp. $L(\mu)$) we have $g_{\Lambda-\mu} c_i^+(\mu) = c_i^+(\Lambda)$. The elements $g_{\Lambda-\mu}$ are units modulo I and hence modulo J. For each simple root α_i, we now fix Λ such that $L(\Lambda)_{\Lambda-\alpha_i} \neq 0$ and set $c_i^+ = c_i^+(\Lambda)$. Then I^+/J is generated by the images of the elements $c_i^+, 1 \leq i \leq n$.

Similarly for each simple root α_i we fix Λ such that $L(\Lambda)_{w_0\Lambda+\alpha_i} \neq 0$, let u^* be the unique element of $L(\Lambda)^*_{-w_0\Lambda-\alpha_i}$ such that $u^*(e_i v_{w_0\Lambda}) = 1$ and set $c_i^- = c_{u^*, v_{w_0\Lambda}}$. Then we have

LEMMA: *The ideal I/J is generated by the images of the elements $c_i^{\pm}$ $1 \le i \le n$.*

2.6. Let $x_1, \ldots, x_m$ be normal elements of a $\mathbb{C}$-algebra R and $I = \sum x_i R$. Suppose that $\overline{R} = R/I$ is commutative and $I^2 = 0$, so that I is an $\overline{R} - \overline{R}$ bimodule. Suppose also that there are nontrivial automorphisms σ_i of $\overline{R}$ such that $x_i r = \sigma_i(r) x_i$ for $i = 1, \ldots, m$. If $\chi \colon \overline{R} \to \mathbb{C}$ is a $\mathbb{C}$-algebra map, let m_χ be the ideal of R which is the preimage of $\ker \chi$ under the canonical map $R \to \overline{R}$.

LEMMA: *If $\operatorname{mult}(m_{\chi'}, m_\chi) > 0$ then one of the following holds:*

(a) $\chi = \chi'\sigma_i$ *for some i and*

$$\operatorname{mult}(m_{\chi'}, m_\chi) \le |\{i | \chi = \chi'\sigma_i\}|$$

(b) $\chi = \chi'$, $I \subseteq m_\chi^2$ *and* $\operatorname{mult}(m_\chi, m_\chi) = \dim m_\chi / m_\chi^2$.

Proof: Suppose first that $\chi \neq \chi'$. Since the images of m_χ and $m_{\chi'}$ in $\overline{R}$ are distinct, and all links in $\overline{R}$ are trivial, we have

$$m_\chi \cap m_{\chi'} = m_{\chi'} m_\chi + I.$$

Also for all $r, s \in \overline{R}$ and $1 \le i \le m$, $m_{\chi'} m_\chi$ contains the elements $(s - \chi'(s))x_i$ and $x_i(r - \chi(r)) = (\sigma_i(r) - \chi(r))x_i$.

If $\chi(r) \neq \chi'\sigma_i(r)$ for some $r \in \overline{R}$, let $s = \sigma_i(r)$, then $m_{\chi'} m_\chi$ contains

$$((s - \chi'(s)) - (\sigma_i(r) - \chi(r)))x_i = (\chi(r) - \chi'(\sigma_i(r)))x_i$$

and hence $x_i \in m_{\chi'} m_\chi$.

On the other hand, if $\chi = \chi'\sigma_i$, then since $\ker \chi'$ has codimension one in $\overline{R}$ we see that

$$R x_i = \overline{R} x_i = (\mathbb{C} + \ker \chi')x_i \subseteq \mathbb{C}\, x_i + m_{\chi'} m_\chi.$$

Therefore

$$I \subseteq \sum_{\chi = \chi'\sigma_i} \mathbb{C}\, x_i + m_{\chi'} m_\chi$$

whence

$$m_\chi \cap m_{\chi'} = m_{\chi'} m_\chi + \sum_{\chi = \chi'\sigma_i} \mathbb{C} x_i$$

and thus leads to the conclusion expressed in (a).

Finally if $\chi' = \chi$ then, since $\sigma_i \neq 1$, the above argument gives $x_i \in m_\chi^2$ for each i, and so $I \subseteq m_\chi^2$. This gives the conclusion in (b).

2.7. Suppose H is a bialgebra, C a subcoalgebra of $H^\star$, and $g, h \in G(C)$. We set $m_g = \operatorname{Ker} g$, and

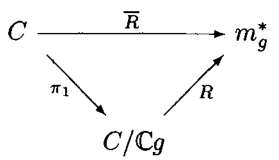
$$\mathbb{C}g \wedge_C \mathbb{C}h = \Delta_C^{-1}(g \otimes C + C \otimes h).$$

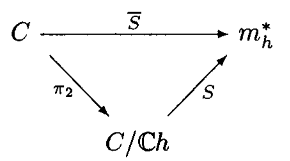
When $C = H^\star$ we simply write $\wedge$ for $\wedge_C$. If $x \in C$ is such that $\Delta(x) = g \otimes x + x \otimes h$, we say that x is (g,h)-primitive. We denote the set of (g,h)-primitive elements of C by $P_{g,h}(C)$. The following result is implicit in [CM1, Proposition 1.1]. For the convenience of the reader we give a self-contained proof, based on [Sw, Proposition A.4].

LEMMA: *For $g, h \in G(H^\star)$ there is a natural isomorphism of vector spaces*

$$\frac{\mathbb{C}g \wedge \mathbb{C}h}{\mathbb{C}g + \mathbb{C}h} \cong \left(\frac{m_g \cap m_h}{m_g m_h}\right)^*.$$

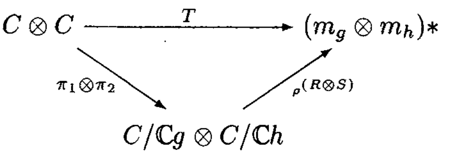
Proof: Let $C = H^\star$. The pairing $C \times H \longrightarrow \mathbb{C}$ induces linear maps $\overline{R}$: $C \longrightarrow m_g^*$ and $\overline{S}$: $C \longrightarrow m_h^*$, with $\operatorname{Ker}\overline{R} = \mathbb{C}g$ and $\operatorname{Ker}\overline{S} = \mathbb{C}h$. Thus $\overline{R}$ and $\overline{S}$ induce injective maps R, S making the following diagrams commute:





where π_1, π_2 are the natural maps.

By [Sw, Proposition A.2] there is a natural inclusion ρ: $m_g^* \otimes m_h^* \longrightarrow (m_g \otimes m_h)^*$. Let T: $C \otimes C \longrightarrow (m_g \otimes m_h)^*$ be the linear map induced by the pairing $(C \otimes C) \times (H \otimes H) \longrightarrow \mathbb{C}$. Then we have another commutative diagram



Since $\rho(R \otimes S)$ is injective, it follows that

$$\begin{aligned} \mathbb{C}g \wedge \mathbb{C}h &= \Delta^{-1}(\operatorname{Ker} \pi_1 \otimes \pi_2) \\ &= \Delta^{-1}(\operatorname{Ker} T) \\ &= \{f \in C \mid \Delta f(m_g \otimes m_h) = 0\} \\ &= \{f \in C \mid f(m_g m_h) = 0\}. \end{aligned}$$

Thus an element f of $\mathbb{C}g \wedge \mathbb{C}h$ may be regarded as a functional on H which vanishes on $m_g m_h$, and the isomorphism is obtained by restricting f to $m_g \cap m_h$.

2.8. Proof of Theorem 2.2: Write $m(\chi', \chi)$ for $\text{mult}(m_{\chi'}, m_\chi)$. We apply Lemma 2.6 to the algebra $R = A/J$ and I the ideal of R generated by the images of the elements $c_i^{\pm}$ for $1 \leq i \leq n$. Note that $I^2 = 0$ and $\overline{R} = R/I \cong \mathbb{C}[H]$ is commutative. Suppose that $w^* \in L(\Lambda)^*_{-\Lambda+\alpha_i}, g \in L(\Lambda')^*_{-\gamma}$ and $v \in L(\Lambda')_\gamma$ are such that $w^*(f_i v_\Lambda) = g(v) = 1$ and $c_i^+ = c_{w^*, v_\Lambda}$. If we denote the images of c_i^+ and $c_{g,v}$ in R by x, c respectively, then by Lemma 2.3(a)

$$cx = q^{(\Phi+\alpha_i, \gamma)} xc.$$

Thus if $\overline{c}$ is the image of c in $\overline{R}$ we have $x\overline{c} = \sigma_i^+(\overline{c})x$, where $\sigma_i^+ \in \operatorname{Aut} \overline{R}$ is defined by $\sigma_i^+(\overline{c}) = q^{-(\Phi+\alpha_i, \gamma)}\overline{c}$. Using Remark 1.8 we see that $\chi' t_i^{-1} = \chi' \sigma_i^+$.

Similarly if y is the image of c_i^- in R, the left and right bimodule structures of the ideal yR are related by $y\overline{c} = \sigma_i^-(\overline{c})y$, for $\overline{c}$ as above, where σ_i^- is defined by $\sigma_i^-(\overline{c}) = q^{(\Phi-\alpha_i, \gamma)}\overline{c}$. It follows from Remark 1.8 that $\chi' s_i^{-1} = \chi' \sigma_i^-$.

Therefore if $m(\chi', \chi) > 0$, Lemma 2.6 implies that either $\chi' = \chi$ and $m(\chi', \chi) = n$, or $\chi' \chi^{-1} \in \Omega$, and $m(\chi', \chi) \leq |\{g \in \Omega | \chi' \chi^{-1} = g\}|$. By Proposition 1.9, we can only have $\chi' = \chi s_i$ for at most one i, and $\chi' = \chi t_j$ for at most one j. Conversely, if both these conditions hold, then $e_i \chi$ and $f_j t_j \chi$ are (χ', χ)-primitive elements of A^* whose images are linearly independent mod $\mathbb{C}\chi' + \mathbb{C}\chi$. Thus Lemma 2.7 shows that $m(\chi', \chi) = 2$. Similarly if $\chi' = \chi s_i$ and $\chi' \neq \chi t_j$ for any j, then $m(\chi', \chi) \leq 1$, and since $e_i \chi$ is a nontrivial (χ', χ)-primitive, we have equality. The remaining case where $\chi' = \chi t_i$ and $\chi \neq \chi' s_j$ for any j is handled in the same way.

The next result follows from Lemma 2.7 and the proof of Theorem 2.2.

Corollary: *If $g \in G(A^*), g \neq 1$ then any $(g, 1)$ primitive element x in A^* can be written in the form*

$$x = a(g-1) + \sum_{i=1}^{n} b_i e_i = \sum_{i=1}^{n} c_i f_i t_i$$

where $a, b_i, c_i \in \mathbb{C}$, $b_i = 0$ unless $g = s_i$ and $c_i = 0$ unless $g = t_i$.

Remark: Theorem 2.2 implies that there are n linearly independent $(1,1)$-primitives in the Hopf dual of $\mathbb{C}_{q,p}[G]$. In the case where q is an indeterminate a description of the Hopf dual of $\mathbb{C}_q[G]$ is given in [J, Proposition 9.4.9].

3. The coradical filtration and Hopf algebra maps

3.1 We first determine the coradical filtration on D and U. More generally let Z be any subgroup of $Z_L, \overline{D} = D/\nabla(Z)$ and denote images in $\overline{D}$ by the overbar. Then $\overline{D}$ has a triangular decomposition $\overline{D} = \overline{D}^- \otimes \overline{D}^0 \otimes \overline{D}^+$. We set $B = \overline{D}^- \otimes \overline{D}^0$, $C = \overline{D}^0 \otimes \overline{D}^+$, $B(0) = C(0) = \overline{D}^0$, $B(1) = \sum \overline{f}_i \overline{D}^0$, $C(1) = \sum \overline{e}_i \overline{D}^0$, $B(m) = B(1)^m$ and $C(m) = C(1)^m$. Then $B = \bigoplus_{m \geq 0} B(m)$ and $C = \bigoplus_{m \geq 0} C(m)$ are graded bialgebras. Note that $\overline{D} = B \otimes_{\overline{D}^0} C$. Set $\overline{D}(m) = \sum_{i+j=m} B(i) \otimes_{\overline{D}^0} C(j)$. Then $\overline{D} = \bigoplus_{m \geq 0} \overline{D}(m)$ is a graded coalgebra. For $m > 0$ set $\overline{D}^m = \bigoplus_{n \leq m} \overline{D}(n)$.

It is easily seen that $\overline{D}^m = (\overline{D}^1)^m$, so $\overline{D}^1$ generates $\overline{D}$ as an algebra. Also $\Delta \overline{D}^1 \subseteq \overline{D}^1 \otimes \overline{D}^0 + \overline{D}^0 \otimes \overline{D}^1$. By [M, Lemma 5.5.1], $\{\overline{D}^m\}$ is a coalgebra filtration of $\overline{D}$ and $\overline{D}^0$ contains the coradical of $\overline{D}$, that is the sum of the simple subcoalgebras of $\overline{D}$. Since $\overline{D}^0$ is spanned by group-like elements, $\overline{D}^0$ equals the coradical of $\overline{D}$, and $\overline{D}$ is pointed. The nth term $\overline{D}^{(n)}$ of the coradical filtration of $\overline{D}$ is defined recursively by setting

$$\overline{D}^{(n)} = \Delta^{-1}(\overline{D} \otimes \overline{D}^{(n-1)} + \overline{D}^0 \otimes \overline{D}).$$

The coradical filtrations of B, C are defined similarly.

THEOREM: *For all $m \geq 0$, $\overline{D}^{(m)} = \overline{D}^m$.*

Proof: We first make some reductions as in the proof of [CM3, 3.7]. By [CM3, Lemmas 2.2 and 2.3] it is enough to show that $B^{(1)} = B(0) \oplus B(1)$ and $C^{(1)} = C(0) \oplus C(1)$. To prove the claim about C, it is enough by [M, Theorem 5.4.1] to show that any $(g, 1)$ primitive element x of C is contained in $C(0) \oplus C(1)$.

Let $U^{\geq} = U_q(b^+)$, and let $\phi: C \longrightarrow U^{\geq} = U^0 \otimes U^+$ be the natural map. Also let $C = \bigoplus_{\alpha \in Q} C_\alpha$, $U^{\geq} = \bigoplus U_\alpha$ be the Q-gradings defined as in 1.8. We may assume $x \in C_\alpha$. If $\alpha = 0$, then $x \in C_0 = C(0)$, so we can assume in addition that $\alpha > 0$. Arguing as in [CM3, 3.7], this implies that $x \in \overline{D}^+_\alpha$. Therefore $y = \phi(x) \in U^+_\alpha$ is $(\phi(g), 1)$ primitive. However $\Delta(y) = y \otimes 1 + k_\alpha \otimes y$ mod terms in $\sum_{\substack{\beta+\gamma=\alpha \\ \beta \neq 0, \alpha}} U^{\geq}_\beta \otimes U^{\geq}_\gamma$. This forces $\phi(g) = k_\alpha \neq 1$. Hence by Corollary 2.8 $y = \lambda e_i$ for some i and $\lambda \in \mathbb{C}$. Since the restriction of ϕ to $\overline{D}^+$ is an isomorphism by Theorem 1.8 we have $x = \lambda \overline{e}_i \in C(1)$ as required.

3.2 Let K be the ideal of D generated by $e_1, \dots, e_n$ and $f_1, \dots, f_n$. Note that $s_i t_i - 1 \in K$ for all i. Also K is a Hopf ideal and D/K is isomorphic to a group algebra.

THEOREM: *If M is any bi-ideal of D, then either $K \subseteq M$ or $M^\dagger \subseteq Z_Q$ and $M = \nabla(M^\dagger)$. In particular M is a Hopf ideal.*

Proof: The proof is similar to the proof of [CM3, Theorem 3], but we repeat some of the details for the convenience of the reader.

Suppose first that we can find an element $s_\lambda t_\mu$ in $M^\dagger \backslash Z_Q$. Then there is a simple root α_j such that $(\Phi_- \lambda - \Phi_+ \mu, \alpha_j) \notin \frac{4\pi i \mathbb{Z}}{\hbar}$. Since M contains $(s_\lambda t_\mu - 1)e_j$ and $e_j(s_\lambda t_\mu - 1)$ it follows easily that $e_j \in M$. Since $e_j f_j - f_j e_j \in M$ we then get $s_j t_j \in M^\dagger$. We can repeat this argument whenever $a_{ij} \neq 0$ to get $e_i, f_i \in M$. By the connectedness of the Dynkin diagram, we have $K \subseteq M$.

Now suppose that $M^\dagger \subseteq Z_Q$, and let $^-$ denote the natural map $^-\colon D \longrightarrow D/\nabla(M^\dagger) = \overline{D}$. If $\overline{M} \neq 0$ then by [M, Theorem 5.3.1], $\overline{M} \cap \overline{D}^{(1)} \neq 0$. From the description of $\overline{D}^{(1)}$ given in Theorem 3.1, and the fact that $\overline{M}$ is Q-graded, we get $\overline{M} \cap \overline{D}^0 \neq 0$, $\overline{M} \cap \overline{e}_i \overline{D}^0 \neq 0$ or $\overline{M} \cap \overline{f}_i \overline{D}^0 \neq 0$. However $\overline{M} \cap \overline{D}^0 = 0$ by construction, and we may assume $\overline{M} \cap \overline{e}_i \overline{D}^0 \neq 0$. If $\sum_j \lambda_j \overline{e}_i g_j \in \overline{M}$ where the g_j are distinct group-likes and the coefficients λ_j are non-zero, we obtain $\sum \lambda_j (\overline{e}_i g_j \otimes g_j + \overline{s}_i g_j \otimes \overline{e}_i g_j) \in \overline{M} \otimes \overline{D} + \overline{D} \otimes \overline{M}$. However the images of $\overline{D} \otimes e_i \overline{D}^{(0)}$ and $\overline{D} \otimes \overline{D}^{(0)}$ in $D/M \otimes D/M$ have zero intersection, and the images of the group-like elements g_j are linearly independent in D/M, so this gives $\overline{e}_i g_j \in \overline{M}$ for all j. Since g_j is a unit we have $\overline{e}_i \in \overline{M}$. As in the first part of the proof, this implies $s_i t_i \in M^\dagger$, a contradiction.

3.3 We determine the Hopf algebra maps between multiparameter Drinfeld doubles. However there is a minor point which we take care of first. As noted in [HLT, 2.1] the isomorphism class of $D_{q,p^{-1}}(g)$ depends only on the cohomology class $[\widetilde{p}]$ in $H^2(L \times L, \mathbb{C}^*)$ of the 2-cocycle $\widetilde{p}\colon (L \times L) \times (L \times L) \longrightarrow \mathbb{C}^*$ defined by

$$\widetilde{p}((\lambda, \mu), (\lambda', \mu')) = p(\lambda, \lambda')p(\mu, \mu')^{-1}.$$

Of course $[\widetilde{p}]$ depends only on the image $[p]$ of p in $H^2(L, \mathbb{C}^*)$, but in certain cases it is possible for a non-identity element of $H^2(L, \mathbb{C}^*)$ to yield a trivial deformation. The reason behind this is that $D_q(g)$ is graded by the subgroup

$$\widetilde{L} = \{(\lambda, \mu) \in L \times L | \lambda + \mu \in Q\}$$

of $L \times L$.

Set $Z_\hbar = \{u \in \Lambda^2 h | u(L \times L) \subseteq \frac{4\pi i \mathbb{Z}}{\hbar}\}$ and denote the set of multiplicatively antisymmetric $n \times n$ complex matrices by $\mathcal{H}$. If $u \in \Lambda^2 h$ and p: $L \times L \longrightarrow \mathbb{C}^*$ is defined, as usual by $p(\lambda, \mu) = \exp(-\frac{\hbar}{4} u(\lambda, \mu))$, there are isomorphisms $H^2(L, \mathbb{C}^*) \cong \mathcal{H} \cong \Lambda^2 h / Z_\hbar$, under which the cohomology class of p corresponds to the image of u, [HLT, Theorem 2.7].

Now set $\widehat{Z}_\hbar = \{u \in \Lambda^2 h | u(L \times Q) \subseteq \frac{4\pi i \mathbb{Z}}{\hbar}\}$. Since $Q \subseteq L$, we have $Z_\hbar \subseteq \widehat{Z}_\hbar$. If $u \in \widehat{Z}_\hbar$ and p is defined as above we have $p(\lambda, \mu) = \pm 1$ for $\lambda \in L$, $\mu \in Q$. Using this it follows that $\widetilde{p}(\alpha, \beta) = \pm 1$ for $\alpha, \beta \in \widetilde{L}$. Thus if $\omega_1, \ldots, \omega_m$ is a basis for the free abelian group $\widetilde{L}$, then the multiplicative matrix γ corresponding to the restriction of $\widetilde{p}$ to $\widetilde{L} \times \widetilde{L}$ has entries $\gamma_{ij} = \widetilde{p}(\omega_i, \omega_j)/\widetilde{p}(\omega_j, \omega_i) = 1$. Hence the image of $\widetilde{p}$ in $H^2(\widetilde{L}, \mathbb{C}^*)$ is trivial. Since $D_q(g)$ is graded by $\widetilde{L}$ we have established the following result.

LEMMA: *The isomorphism class of the Hopf algebra $D_{q,p^{-1}}(g)$ depends only on the image of u in $\Lambda^2 h / \widehat{Z}_\hbar$.*

We remark that if L/Q is cyclic, then using elementary divisor theory and the fact that u is skew-symmetric it follows that $Z_\hbar = \widehat{Z}_\hbar$. In this case the foregoing remarks are unnecessary. Furthermore, if L/Q is not cyclic then from the list of fundamental groups contained in [H, page 68 and exercise 4, page 71], it follows that G is simply connected of type $D_{2\ell}$, $\ell \geq 2$.

3.4 Let (a'_{ij}) be a second Cartan matrix associated to the simple Lie algebra g' of rank m and suppose $\{d'_i\}$ are relatively prime positive integers such that $(d'_i a'_{ij})$ is symmetric. Let G' be a connected algebraic group with Lie algebra g' and character group L'. If Q' is the root lattice of g, then $\widehat{Z}'_\hbar$ is defined by $\widehat{Z}'_\hbar = \{u' \in \Lambda^2 h' | u'(L' \times Q') \subseteq \frac{4\pi i \mathbb{Z}}{\hbar}\}$. Let $p' \in \Lambda^2 h' / \widehat{Z}'_\hbar$ and choose $u' \in \Lambda^2 h'$ such that $p'(\lambda, \mu) = \exp(-\frac{\hbar}{4} u'(\lambda, \mu))$, for $\lambda, \mu \in L$. Set $\Phi' = -{}^t u'$. We consider a second multiparameter Drinfeld double $D' = D'_{q,(p')^{-1}}(g')$ with generators e'_i, f'_i s'_λ, t'_λ, $\lambda \in L'$ and relations as in 1.6 with e_i, f_i, s_λ, t_λ, Φ replaced by e'_i, f'_i, s'_λ, t'_λ, Φ'.

First we describe some Hopf algebra automorphisms of D. Let $N = (\mathbb{C}^*)^n$ and for $a = (a_1, \ldots, a_n) \in N$ define the diagonal automorphism ϕ_a: $D \longrightarrow D$ by

$$\phi_a(s_\lambda) = s_\lambda, \quad \phi_a(t_\lambda) = t_\lambda, \quad \lambda \in L,$$
$$\phi_a(e_i) = a_i e_i, \quad \phi_a(f_i) = a_i^{-1} f_i, \quad i = 1, \ldots, n.$$

Let $\mathbf{n} = \{1, \ldots, n\}$. An injective map $\sigma: \mathbf{n} \longrightarrow \mathbf{m}$ is an inclusion of Dynkin diagrams if $(\alpha_i, \alpha_j) = (\alpha'_{\sigma(i)}, \alpha'_{\sigma(j)})$ for all $i, j \in \mathbf{n}$. If this is the case then $\alpha_i \longrightarrow \alpha_{\sigma(i)}$ induces a map σ: $h^* \longrightarrow (h')^*$ which carries Q into Q'. Suppose that $\sigma(L) \subseteq L'$. If $u' \in \Lambda^2 h'$, we denote by ${}^\sigma u'$ the pullback of u' to $\Lambda^2 h$, defined by

$${}^\sigma u'(\lambda, \mu) = u'(\sigma\lambda, \sigma\mu).$$

The pullback ${}^\sigma p'$ of $p' \in \Lambda^2 h'/\widehat{Z}'_{\hbar}$ is defined similarly.

Suppose $u = {}^\sigma u'$, and set $p'(\lambda, \mu) = q^{\frac{1}{2}u'(\lambda,\mu)}$ for $\lambda, \mu \in L'$. Then σ induces a Hopf algebra map σ: $D_{q,p^{-1}}(g) \longrightarrow D'_{q,(p')^{-1}}(g')$ such that

$$\sigma(t_\lambda) = t'_{\sigma(\lambda)}, \quad \sigma(s_\lambda) = s'_{\sigma(\lambda)}, \quad \lambda \in L,$$
$$\sigma(e_i) = e'_{\sigma(i)}, \quad \sigma(f_i) = f'_{\sigma(i)}, \quad i = 1, \ldots, n.$$

The case where ϕ: $D \longrightarrow D'$ is a Hopf algebra map such that $e_1, \ldots, e_n, f_1, \ldots, f_n \in \operatorname{Ker}\phi$ is rather uninteresting, so we concentrate on the other possibility allowed by Theorem 3.2.

THEOREM: *Let ϕ: $D \longrightarrow D'$ be a $\mathbb{C}$-linear Hopf algebra map such that $(\operatorname{Ker}\phi)^\dagger \subseteq Z_Q$. Then we can write ϕ uniquely in the form $\phi = \sigma\phi_a$, where ϕ_a is a diagonal automorphism and σ an inclusion of Dynkin diagrams such that $\sigma(L) \subseteq L'$ and $p = {}^\sigma p'$. In particular ϕ is injective.*

Proof: For each $i \in \mathbf{n}, \phi(e_i)$ is a nontrivial $(\phi(s_i), 1)$ primitive in D'. Hence by Corollary 2.8,

$$\phi(e_i) = \sum_{j=1}^{m} a_j e'_j + b_j f'_j t'_j + c(\phi(s_i) - 1)$$

where $a_j = 0$ unless $\phi(s_i) = s'_j$ and $b_j = 0$ unless $\phi(s_i) = t'_j$. Applying ϕ to the equation $s_i e_i s_i^{-1} = q^{(\alpha_i, \alpha_i)} e_i$ shows that $\phi(e_i) = a_j e_j$ for some $j \in \mathbf{m}$ and some nonzero a_j. Thus there is a map σ: $\mathbf{n} \longrightarrow \mathbf{m}$ such that $\phi(s_i) = s'_{\sigma(i)}$ and $\phi(e_i) = a_{\sigma(i)} e'_{\sigma(i)}$. By Corollary 1.9, σ is injective. In addition we have $(\alpha_i, \alpha_i) = (\alpha_{\sigma(i)}, \alpha_{\sigma(i)})$, thus $d_i = d_{\sigma(i)}$. Similarly there is an injective map τ: $\mathbf{n} \longrightarrow \mathbf{m}$ such that $\phi(t_i) = t_{\tau(i)}$ and we have $\phi(f_i) = c_{\tau(i)} f_{\tau(i)}$ for some nonzero $c_{\tau(i)}$. Now applying ϕ to the equation

$$(e_i f_i - f_i e_i) = \widehat{q}_i (s_i - t_i^{-1})$$

shows that $\sigma = \tau$ and $a_{\tau(i)}c_{\tau(i)} = 1$ for all i. Note also that the unique extension of σ to a map $\sigma\colon h^* \longrightarrow (h')^*$ satisfies $\sigma(L) \subseteq L'$ since ϕ maps group-like elements in D to group-like elements in D'

Next we claim that

$$(*)_\epsilon \qquad (\Phi_\epsilon\lambda, \alpha_j) - (\Phi'_\epsilon\sigma(\lambda), \alpha_{\sigma(j)}) \in 4\pi i\mathbb{Z}/\hbar$$

for $\lambda \in L, j = 1, \ldots, n$ and $\epsilon = \pm$.

For, applying ϕ to the equation $s_\lambda e_j s_{-\lambda} = q^{-(\Phi_-\lambda,\alpha_j)}e_j$ we obtain

$$q^{(\Phi_-\lambda,\alpha_j)-(\Phi'_-\sigma(\lambda),\alpha_{\sigma(j)})} = 1$$

which yields $(*)_-$. Similarly applying ϕ to the equation $t_\lambda e_j t_{-\lambda} = q^{(\Phi_+\lambda,\alpha_j)}e_j$ yields $(*)_+$.

Subtracting $(*)_-$ from $(*)_+$ gives $(\lambda, \alpha_j) = (\sigma\lambda, \alpha_{\sigma(j)})$ since $\hbar \notin i\pi\mathbb{Q}$. Thus σ is an inclusion of Dynkin diagrams. In addition

$$u(\lambda, \alpha_j) - u'(\sigma(\lambda), \alpha_{\sigma(j)}) = (\Phi\lambda, \alpha_j) - (\Phi'\sigma\lambda, \alpha_{\sigma(j)}) \in 4\pi i\mathbb{Z}/\hbar$$

which shows that $p = {}^{\sigma}p'$.

The group Γ of automorphisms of the Dynkin diagram acts naturally on $H = \Lambda^2 h/\widehat{Z}_\hbar$. We denote the stabilizer of p by Γ_p.

COROLLARY: *Fix q, g and L. If $p, p' \in H$ then $D_{q,p^{-1}}(g) \cong D_{q,(p')^{-1}}(g)$ if and only if $p = {}^{\sigma}p'$ for some $\sigma \in \Gamma$. In addition $\mathrm{Aut}_{\mathrm{Hopf}}\, D_{q,p^{-1}}(g) = N\Gamma_p$.*

3.5 We briefly consider Hopf algebra maps between multiparameter enveloping algebras. Here additional complications arise. Suppose $D = D_{q,p^{-1}}(g)$ and $D' = D_{q,(p')^{-1}}(g')$ are multiparameter Drinfeld doubles and that $\sigma\colon D \longrightarrow D'$ is induced by an inclusion of Dynkin diagrams such that $p = {}^{\sigma}p'$. Let R and R' be the radicals of D and D' and set $U = D/R, U' = D'/R'$. In order for σ to induce a map from U to U' we require $\sigma(Z_L) \subseteq Z_{L'}$. Now the condition that $s_\lambda t_\mu$ belongs to Z_L is that

$$(1) \qquad u(\lambda - \mu, \beta) - (\lambda + \mu, \beta) \in 4\pi i\mathbb{Z}/\hbar \quad \text{for all } \beta \in L$$

and this yields only that

$$u'(\sigma\lambda - \sigma\mu, \sigma\beta) - (\sigma\lambda + \sigma\mu, \sigma\beta) \in 4\pi i\mathbb{Z}/\hbar \quad \text{for all } \beta \in L.$$

Thus σ is unlikely to induce a map from U to U' unless $\sigma(L) = L'$.

There is another minor point. Suppose that (1) holds and that $u'' = u + u'$ where $u' \in Z_\hbar$. Since $u'(\lambda - \mu, \beta) \in 4\pi i\mathbb{Z}/\hbar$ for all $\beta \in L$, equation (1) holds with u replaced by u''. Thus the definition of Z_L depends only on the image of u in $\Lambda^2 h/Z_\hbar = H^2(L, \mathbb{C}^*)$. However, it is not clear that the same can be said about the image of u in $\Lambda^2 \underline{h}/\widehat{Z}_\hbar$.

We can avoid this difficulty by restricting our attention to the case where L/Q is cyclic. Then imitating the arguments in 3.4 we have

THEOREM: *Assume G is not simply connected of type $D_{2\ell}, \ell \geq 2$. Then for $p, p' \in H^2(L, \mathbb{C}^*)$, $U_{q,p^{-1}}(g) \cong U_{q,(p')^{-1}}(g)$ if and only if $p = {}^\sigma p'$ for some $\sigma \in \Gamma$. Moreover*

$$\mathrm{Aut}_{\mathrm{Hopf}}\, U_{q,p^{-1}}(g) \cong N\Gamma_p.$$

For the one-parameter case this recovers [T1, Theorem 2.1], see also [CM, 3, Corollary 4.3]. More details of the argument used in [T1] may be found in [T2]. Note, however, that when q is a root of unity there are more skew primitives than those given in [T2, Lemma 1.2.7]. Thus the group of Hopf algebra automorphisms of $U_q(\underline{g})$ is still unknown when q is a root of unity. A version of Theorem 3.5 for $g = \mathrm{sl}(n)$ will appear in [C].

A related result is obtained in [B], which describes embeddings of quantum groups, in the sense of Hopf algebra surjections of their function algebras. Any such embedding can be obtained from those defined by an embedding of Dynkin diagrams by means of the adjoint action of the maximal torus of the embedded group.

3.6 We show how the results of this section can be extended to Drinfeld doubles defined over arbitrary fields of characteristic zero. If H is a Hopf algebra defined over a field K, and $K \subseteq F$ is a field extension, we write H^F for $H \otimes_K F$.

LEMMA: *If g and h are group-like elements of H then $P_{g,h}(H) \otimes_K F = P_{g,h}(H^F)$.*

Proof: Left to the reader.

Now with the notation as in 1.1, suppose that G is the connected simple algebraic group with Lie algebra g and character group L. Let F be a field of characteristic zero, and p: $L \times L \longrightarrow F^*$ an antisymmetric bicharacter. We write D for the Hopf algebra over F generated as an algebra by group-like elements

s_λ, t_λ ($\lambda \in L$), $(s_{\alpha_i}, 1)$-primitive elements e_i and $(1, t_{\alpha_i}^{-1})$-primitive elements f_i subject to the requirements that for all $\lambda \in L$ and $1 \leq i, j \leq n$

$$\begin{aligned} e_i f_j - f_j e_i &= \delta_{ij} \widehat{q}_i (s_{\alpha_i} - t_{\alpha_i}^{-1}), \\ s_\lambda e_j s_{-\lambda} &= q^{(\lambda, \alpha_j)} p(\lambda, \alpha_j)^{-2} e_j, \\ s_\lambda f_j s_{-\lambda} &= q^{-(\lambda, \alpha_j)} p(\lambda, \alpha_j)^{2} f_j, \\ t_\lambda e_j t_{-\lambda} &= q^{(\lambda, \alpha_j)} p(\lambda, \alpha_j)^{2} e_j, \\ t_\lambda f_j t_{-\lambda} &= q^{-(\lambda, \alpha_j)} p(\lambda, \alpha_j)^{-2} f_j, \\ (ad'_r(e_i))^{1-a_{ij}}(e_j) &= 0, \quad i \neq j, \\ (ad_r(f_i))^{1-a_{ij}}(f_j) &= 0, \quad i \neq j, \end{aligned}$$

and the elements $\{s_\lambda t_\mu | (\lambda, \mu) \in L \times L\}$ form a group isomorphic to $L \times L$.

In this situation, we claim there is a description of the coradical filtration of D which is analogous to that given in Theroem 3.1. By the proof of Theorem 3.1, it is enough to describe the $(g, 1)$-primitive elements of D for $g \neq 1$. The lemma allows us to assume that F is the field generated over $\mathbb{Q}$ by q and the image of p. In this case since F is countable, there is a $\mathbb{Q}$-linear embedding of F into $\mathbb{C}$. Again, by the lemma we may replace F by $\mathbb{C}$. Then there exists $\hbar \in \mathbb{C}^*$ and $u \in \Lambda^2 h$ such that $q = \exp(-\hbar/2)$ and $p(\lambda, \mu) = \exp(-\frac{\hbar}{4} u(\lambda, \mu))$. Thus we have reduced our claim to the case considered in 1.6 and Theorem 3.1.

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